UNIT DISTINCTION AS A PRE-REQUISITE FOR MULTIPLICATIVE REASONING: A CASE STUDY OF ADAM'S UNIT TRANSFORMATION

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We address the question: How can a student's conceptual transition, from attending only to singleton units (1s) given in multiplicative situations to distinguishing composite units made of such 1s, be explained? We analyze a case study of one fourth grader (Adam, a pseudonym) during the course of a video recorded cognitive interview. Adam's case provides a glimpse into this process, as he proceeded from solving a prompt-less version of the multiplicative situation to two versions that gave gradually more explicit hints for those two types of units. We postulate an inferred process in which Adam reflected on the relationships between his counting actions (of cubes and towers) and the effect of those actions being different numbers pertaining to different units. We discuss the theoretical importance of explaining this transition and its practical implication of understanding whole number multiplication (e.g., array).

Keywords: Reasoning and Proof, Learning Trajectories, Number Concepts and Operations

In this case study of a fourth-grade student named Adam (pseudonym), we addressed the question: *How can a student's conceptual transition, from attending only to singleton units (1s) given in multiplicative situations to distinguishing composite units made of such 1s, be explained?* Our study is situated within and contributes to the body of research focusing on students' units coordination as a conceptual lens to explain multiplicative reasoning (Steffe, 1992; Norton et al., 2015; Ulrich, 2016). These researchers demonstrated that, to reason multiplicatively, a student needs to coordinate and simultaneously operate on at least two qualitatively distinct types of units: singletons (1s) and composite units comprised of such 1s. Our study further elaborates on this stance in two important ways: (a) pointing to the conceptual prerequisite distinguishing between 1s and composite units plays in then also coordinating them multiplicatively and (b) explaining the conceptual process involved in arriving at this distinction.

Explaining this conceptual transition can contribute to understanding challenges students face when taught multiplication. A student may try to solve an archetypical multiplicative situation consisting of several equal-size groups, such as: How many cubes do you need to build 6 towers, each made of 3 cubes? To solve this problem, the student would need to understand two types of different units are involved: single cubes (1s) and towers comprised of three such cubes each (composite units). This distinction seems necessary also in cases where the units are presented as an array (e.g., 6 rows by 3 columns) with a total of eighteen 1s (e.g., dots). In this case, the student would need to unpack the organization of 18 items into rows and columns as a highly compact form in which she needs to coordinate two different units.

Theoretical and Conceptual Framework

Our framework for this study draws on Piaget's (1985) concept of assimilation, which stresses a learner can only make sense of some information by using available (assimilatory) conceptions. For example, upon reading the problem situation above, a student who is yet to distinguish 1s from composite units may literally see both given numbers (6 and 3) while only "taking in" the number of single cubes (3). Assimilation is thus the starting point for any transition to a new conception, which Piaget (1985) and von Glasersfeld (1995) explained by the mental process of reflective abstraction. Simon et al. (2004) elaborated on this process, in a mechanism they termed *Reflection on Activity-Effect Relationships* (Ref*AER). In a nutshell, Ref*AER consists of four mental steps: 1) setting a goal in the problem situation, 2) calling up and initiating an activity sequence to attain that goal, 3) determining if the effects of the activity match the goal, and if not 4) adjusting the activity sequence as needed. In our example, a student may come to link her or his mental actions (e.g., counting 1s) with the effects of those actions (e.g., noticing that the count of cubes resulted in "3" whereas the count of towers resulted in "6"). Tzur & Simon (2004) further distinguished two stages of mathematical conceptual learning based on the extent to which a learner's new conceptions depends on some prompting or can be independently and spontaneously called up and used by the learner. The scope of this study led us to focus the data analysis on Adam's transition to a prompt-dependent form of distinguishing 1s from composite units

As we noted above, the conceptual transformation on which we focus in this study is rooted in the explanation of multiplicative reasoning in terms of units coordination. Of the six multiplicative schemes that Tzur et al. (2013) identified, our study focuses on the first, called *Multiplicative Double Counting* (mDC). Units coordination that characterizes mDC involves the distribution of items of one composite unit (e.g., three 1s) into the items of another composite unit (e.g., six equal-size units, such as towers made up of 3 cubes each). This coordination of two types of units is essential not only for developing multiplicative reasoning but also fractional reasoning (Norton et al., 2015; Hackenberg, 2013; Steffe & Olive, 2010; Hackenberg & Tillema, 2009). By addressing an initial phase in the conceptual transition from not distinguishing between composite units and 1s that constitute them to ably doing so, this study can help explain how this conceptual basis for coordinating both types of units may come about.

Methods

Setting and Participants

This is a case study focused on examining a conceptual transition – a phenomenon requiring in-depth analysis of the participant's meanings (Stake, 1995) while solving mathematical word problems. A fourth grader (Adam, pseudonym), from a school in a large US city, participated in a cognitive interview to assess his assimilatory scheme for mDC. The school he attended reported a percentage of English Language Learners at 54.2% and students receiving free or reduced lunch at 84.9%.

We chose Adam as a case because of three key reasons. First, he exemplifies the conceptual transition at hand. Second, he is a fluent reader whose mathematical abilities did not seem rooted in difficulties to comprehend the questions. Third, he solved the additive screener problem (see next section) by doubling 7+7=14, then adding 1 more to obtain the correct answer of 15. This solution indicated to us Adam capably distinguished and used composite units and 1s to assimilate and solve additive problem situations. Combined, these three reasons provide a basis for analyzing a transition to a unit distinction required to assimilate (make sense) and solve

multiplicative situations involving 1s and composite units.

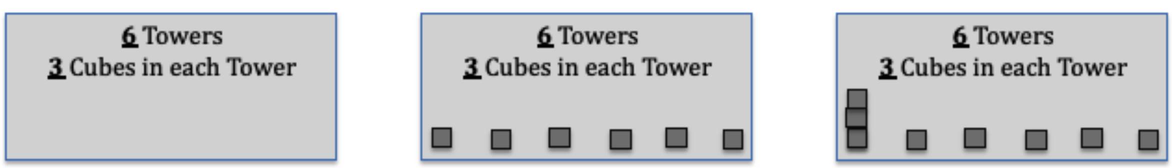
Problem Situations Adam Solved

Our team developed and validated the 5-item assessment, used during the cognitive interview, as a measure of students' mDC scheme (Johnson, et al., 2018). The first item of the measurement, intended as an additive screener question, is followed by four items to measure mDC reasoning. All four mDC items are expressed as word problems and include diagrams to illustrate the given units. Table 1 presents the additive screener and the mDC item (Problem 3 with its four sub-questions) we used in this study of Adam's case. It should be noted that, prior to solving any of those items, Adam literally built a tower of 7 cubes, and then correctly pointed to a diagram of such a tower that was included in the assessment.

Like all four mDC items, Problem 3 begins with a prompt-less version consisting of 4 questions (Figure 1a). Once the child attempts solving that version, the interviewer moves to the next version, which gives a first hint (showing there are 6 composite units) followed by the same four questions (Figure 1b). Finally, the interviewer moves to the third version, which includes a more explicit hint (showing the 6 composite units and the three 1s that comprise the first of them) followed by the same four questions (Figure c). This sequencing of assessment items – progressing from prompt-less to gradually more explicit hints – draws on Tzur's (2007) notion of fine grain assessment.

Table 1: mDC Measurement Word Problems

#	Word Problem
1	For breakfast Ana ate 8 grapes. For lunch, Ana ate 7 grapes. How many grapes did Ana eat in all? Ana ate _______ grapes in all.
3	The picture to the right shows a box. Alex put **6** towers in the box. Alex made each tower with **3** cubes. The numbers on the picture show this. How many **cubes in all** did Alex use to make **6** towers? (fill in the blank): ______



1a: Prompt-less version **1b: First-hint version** **1c: Second-hint version**

Figures 1a, 1b, and 1c: Diagrams Provided Along with Problem 3

Cognitive Interview

The third author of this paper conducted the cognitive interview with Adam. This cognitive interview was one of 40 similar interviews that constituted the last phase of the validation process of the 5-item measure. Accordingly, it required of the interviewer to give no additional support (or probing) beyond what is presented in the written assessment. The protocol for this interview involved first asking Adam if he would like to read the problems on his own or be read to by the researcher. Adam proudly opted for the former. Then, the researcher presented, one at a time, the single page on which each Problem version with its sub-questions is given. Adam read the problem situation and then the sub-questions out loud, while solving each of those questions

as he thought proper without any further assistance from the researcher.

Data Collection and Analysis

The cognitive interview was video-recorded and stored along with Adam's written work for later analysis. Three team members (the PI and two graduate research assistants) individually analyzed the video, while taking extensive notes and identifying key moments they would then use to analyze the data. They met to discuss their notes, identified agreed upon key moments, and then then first author (graduate assistant) transcribed those video segments. After scrutinizing the video and transcriptions line-by-line, we chose Problem 3 for the analysis presented in this paper, as it provides the most compelling evidence of Adam's conceptual transition from not distinguishing to distinguishing between 1s and composite units within a multiplicative situation.

Results

We provide evidence of Adam's transformation of his available conceptions that led him to distinguish between single units (1s) and composite units given in a problem situation involving multiplicative relationships. We first analyze data and offer inferences of plausible conceptual sources of his initial inability to solve a task that provided no hints. Then, we articulate how his work on versions of the problem situation that included hints indicates a transformation in his reasoning. We demonstrate his novel distinction between 1s and composite units, which is a conceptual prerequisite for properly operating on both to solve multiplicative problem situations.

We begin with Adam's work on the prompt-less version of Problem #3, which asked to find the total number of cubes (1s) when given 6 towers (number of composite units) each comprised of 3 cubes (unit rate). Excerpt 1 shows Adam's work on the prompt-less version (see Figure 1 above). He read the problem out loud and began answering each sub-question.

Excerpt 1. Prompt-less questions of Problem 3.

A: (Reads the problem aloud, and rather quickly) "The picture to the right shows a box. Alex puts 6 towers in the box. Alex made each tower with 3 cubes. The numbers on the picture show this." (Here, he looks at the box for 2 seconds, then moves on to reading Question 1: "How many towers …?" and incorrectly writes "3". Then, he reads Question 2 silently: "How many cubes per tower …?" and correctly writes "3.")

R: You wanna read that [Question 3] out loud?

A: (Reads Question 3 silently twice: "How many cubes in all?" He looks around and shakes his head) I don't get this.

R: Okay, you can write I don't know, IDK; that's fine.

A: (Looks back and re-reads Question 3 silently, looks up while thinking how to solve it.)

R: Take your time.

A: (Writes, "I don't know," then reads Question 4 silently: "How many cubes are in 2 towers?" He quickly and incorrectly writes "3.")

R: Good? [This question meant, 'Are you good to move on to the next problem?']

A: (Nods, "Yes.")

Data in Excerpt 1 indicate Adam's assimilation of this prompt-less problem situation did not include distinction among different units. To each of the questions, he essentially responded with "3." Whereas this number is a correct answer to the unit rate, it is incorrect when answering the number of composite units (6 towers), the total of 1s (18 cubes), and even the total of 1s in just

two towers (6 cubes). We infer from those answers that he could think of the 1s given in the problem (3 cubes per tower), while not yet distinguishing the composite units as six individual items, let alone as six items composed of three 1s each. We thus consider Adam's initial response to Question 3 ("I don't know"), about the total of 1s in all six towers to be genuine. Along with his independent effort to re-read this question twice, the data support our claim that he assimilated the entire problem into a conception that is yet to *distinguish composite units within a multiplicative situation*. We emphasize multiplicative situation, as Adam seemed capable of assimilating composite units in his solution to the additive, screener problem (doubling 7+7 to answer 8+7).

Having completed all four questions in the prompt-less version of Problem 3, Adam continued to its next part, which provides the first hint – a diagram showing the bottom cube of each tower (Figure 2). Aside from this hint, he would then have to answer the same four questions. While Adam's was noticeably slower in his reading of this version than his reading of the prompt-less version, he answered the first two questions the same (incorrect) way. He then also began solving Question 3 similarly. Yet, Excerpt 2 shows that a first perturbation seemed to occur as he further thought of the total of cubes in all 6 towers of 3 cubes each.

Excerpt 2. First-prompt questions.

A: (Reads the problem aloud, noticeably slower pace, touching each word with his pen as he reads it.) "The picture to the right shows a box. Alex put 6 towers in the box. You can see the bottom cube of each ..."

R: (interrupts his reading to orient attention to the hint) Do you see them [the bottom cubes]?

A: (Looks at the diagram, touches one of the cubes with his pen to indicate he does, then continues reading to the end of the problem.) "… 6 towers in the box. You can see the bottom cube of each tower. Alex made each tower of 3 cubes. The numbers on the picture show this."

R: So, can you answer this one?

A: Yah (Reads Question 1 silently. Then, he rereads the given problem situation, and incorrectly writes "3" for number of towers. He then reads Question 2 silently and, similarly to his prompt-less response, correctly writes "3" for the number of cubes in each tower. He moves to reading Question 3 silently (twice), then talk to R as if having an "aha" moment): Wait; so the towers are made of 6? There's 6?

R: (Not confirming Adam's realization directly, but instead restates Question 3) So, "How many cubes **in all** did Alex use to make 6 towers?" You can read it again, or look at the picture, what are the towers and the cubes there. The question is how many cubes **in all** Alex used to make 6 towers.

A: (Writes, incorrectly, "0," then also answers Question 4 incorrectly with "12" cubes as the total in only 2 towers of 3 cubes each.)

Data in Excerpt 2 further support our claim that, while solving the first two versions of Problem 3, Adam's assimilatory conception was yet to include a distinction between units of 1 and other units composed of 1s. In spite of explicitly pointing to the diagram presenting the six bottom cubes of each tower, he still incorrectly responded ("3") to the first three questions. Importantly, an event that could become a turning point in his transition occurred when, upon slowly rereading Question 3, he indicated for the first time a recognition of another numerical quantity that constitutes the problem situation ("Wait; so the towers are made of 6? There's 6?").

This recognition seemed supported by his (a) action following the hint (pointing to the six cubes in the diagram), (b) slower reading of all parts of Problem 3, and (c) rereading of Question 3.

The importance of this perturbing experience, seemingly an "aha" moment for Adam, is that he independently identified the other given number, which he overlooked (did not assimilate) in the previous attempts. His answers to Question 3 and Question 4 were still incorrect. However, his solution to the following, last part of Problem 3, which includes a second, more explicit hint, suggests this realization would contribute to his novel recognition and distinction of the other number as a salient feature of the problem situation.

The final part of Problem 3 included a diagram that shows the bottom cube for all 6 towers as well as all 3 cubes that compose the first tower on the left (Figure 3). The purpose of this hint is to provide a student with a more explicit opportunity to assimilate the two types of units that constitute the problem situation. Excerpt 3 presents Adam's first transition to reasoning with both unit types.

Excerpt 3. Second-prompt questions.

A: (Reads the problem situation aloud in slow pace, touching each word with his pen as he reads it.) "The picture to the right shows a box. Alex put 6 towers in the box. You can see the bottom cube of each tower and all 3 cubes of a tower." (He looks at the diagram, tapping the words about the two different types of units involved (in the box) as he reads them to himself. He then moves on to read Question 1 silently, looks back at the diagram, and *correctly writes "6"* in the blank for the number of towers Alex put in the box. He then reads Question 2 aloud while immediately and correctly answering it) "Alex made each tower with 3." (He writes the answer in the blank and moves to Question 3). "How many cubes did Alex use to make 6 towers?" (He looks at the diagram, makes a circle in the air above the completed tower showing 3 cubes, makes another circle in the air above the next cube that is the basis of the second tower, looks to R, makes circles in the air above the other cubes while likely counting them to six, rereads the question silently, looks up to think more about the question, and finally writes "IDK" for I don't know.)

R: That's okay [confirming IDK as a response].

A: (Reads question 4 silently, looks at that diagram again, and *correctly writes the answer "6"* for the number of cubes in just two towers of 3 cubes each.)

Data in Excerpt 3 indicate the two key aspects of our claim: in Adam's work on the version with a second hint, he was (a) finally able to distinguish between the two units while (b) not yet coordinating them multiplicatively when asked about all cubes (1s) in the six towers (composite units) made of 3 cubes each (unit rate). The first aspect is evidenced in correctly answering Question 1 by identifying the 6 towers (composite units) and in correctly answering Question 2 by identifying the 3 cubes (single units) that composed one tower. We inferred from these two correct answers that Adam has begun a transition to a conception that involves the 1s in each composite unit and the composite units made of those 1s as two distinct types of units.

Having just distinguished the two unit types, operating on both to account for all 1s in six composite units seemed beyond Adam's evolving conceptualization (writing "IDK"). On the other hand, operating on both in Question 4 became possible by this conceptualization. We infer he could then consider the first two towers he might build, each with 3 cubes, similarly to how he used 7+7 when solving the additive screener problem.

Adam's progression through all parts of Problem 3 indicates his transition from inability to an initial ability of distinguishing units of 1 from composite units. We postulate that this transition was initially supported by his reflection on the activity of counting the six cubes in the diagram (Figure 2) provided as the first hint that seemed to enable his "aha" moment. This transition seemed further supported by Adam's reflection on the effects of his explicit actions of counting when studying the problem situation with the second hint. Using two distinct actions of circling above the relevant units, he produced two different effects: counting to three cubes in the first tower and counting to six cubes representing the bottom cube of each tower.

That this transition is not trivial can be emphasized in the sequence of his solutions. Before being prompted, Adam seemed to overlook the multiple units "given" in the problem situation – he seemed capable of identifying only one unit (3), which he used throughout his answers for that problem. Once Adam was given the first hint, he experienced a perturbation that could set in motion his distinction between the units involved in the problem. However, this distinction was yet to be made explicit. Thus, he was still not able to operate correctly with both unit types. With the second hint, Adam's unit distinction had taken on the transformation needed to solve the multiplicative situation. Correctly identifying both units – 6 towers/composite units and 3 cubes in each tower/single units – opened the way to also coordinating them at least when solving Question 4 that dealt with only two towers. His distinction seemed too rudimentary for also coordinating the two types of units when the task involved more than two composite units.

Discussion

We see two key contributions of this study. For research and theory building, it demonstrates how a student engaging in solving mDC problem situations with gradual hints can advance from having no distinction of units to beginning to distinguish between composite units and singletons (1s). That is, the study provides a lens through which to understand the prerequisite role that distinguishing those two types of units serves in advancing to multiplicative operations on such units. Simply put, before operating multiplicatively on two different types of units one must explicitly distinguish them. Moreover, the study goes beyond noting these two, before-and-after conceptual markers (Tzur, 2019), to articulating a mental process involved in the transition from the former (no distinction) to the latter ("seeing" composite units apart from the 1s of which they are comprised). Specifically, we pointed out to Adam's reflection on the effects of his counting activities as a plausible way this transition might have come about (Simon et al., 2004). We do not claim every student would make this critical transition in the same way. However, we believe this study provides a sensible explanation of it. In Adam's case, the role this distinction of units played in his operation on both types of units was manifested in his correct solution to Question 4 (6 cubes in 2 towers of 3 cubes each) of the second-hint version of the problem situation.

For practice, our study demonstrates the importance of developing a way of distinguishing and operating on those two types of units as a basis for learning to meaningfully use arrays as a representation of think whole number multiplication. Adam's case of distinguishing the two units that constitute such arrays can inform teachers about ways their students' may reason is such instances. Students who are yet to make such a distinction would likely reason about such arrays as constituted of all 1s (single items). Such students may hear and assimilate "rows" and "columns" similarly to how Adam read and assimilated the two units given in the prompt-less situation, namely, as 1s. Adam's case also illustrates how students may be able to identify and distinguish between 1s and composite units when they are prompted. Importantly, his case also demonstrates how the students may be able to distinguish between 1s and composite units while

not yet being able to operate on those units multiplicatively. We contend that arrays would represent multiplication meaning for students only when they can both make this distinction explicit (as Adam began to do) and operate on the units beyond doubling (as he was yet to do). This information about how the students assimilate and interpret word problems provides a teacher with a conceptual basis to determine where they could go next (i.e., goals for the students' learning) and what activities could be designed to promote the intended advance. Specifically, for students like Adam the goal would be to support their conceptual ability to independently and explicitly distinguish among the units (prompt-less). For students who accomplished such a conceptual ability – the goal would be to promote their multiplicative operations with/on those units.

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